RADII OF COVERING DISKS FOR LOCALLY UNIVALENT HARMONIC MAPPINGS

SERGEY YU. GRAF, SAMINATHAN PONNUSAMY, AND VICTOR V. STARKOV

ABSTRACT. For a univalent smooth mapping f of the unit disk $\mathbb{D}$ of complex plane onto the manifold $f(\mathbb{D})$, let $d_f(z_0)$ be the radius of the largest univalent disk on the manifold $f(\mathbb{D})$ centered at $f(z_0)$ ($|z_0| < 1$). The main aim of the present article is to investigate how the radius $d_h(z_0)$ varies when the analytic function h is replaced by a sense-preserving harmonic function $f = h + \overline{g}$. The main result includes sharp upper and lower bounds for the quotient $d_f(z_0)/d_h(z_0)$, especially, for a family of locally univalent Q-quasiconformal harmonic mappings $f = h + \overline{g}$ on $|z| < 1$. In addition, estimate on the radius of the disk of convexity of functions belonging to certain linear invariant families of locally univalent Q-quasiconformal harmonic mappings of order α is obtained.

1. Introduction and Main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk, and h be a smooth univalent mapping of the unit disk $\mathbb{D}$ onto two-dimensional manifold M. For a point $a \in \mathbb{D}$, we write $d_h(z)$ as the radius of the largest univalent disk centered at $h(a)$ on the manifold M. Here a univalent disk on M centered at $h(a)$ means that h maps an open subset of $\mathbb{D}$ containing the point a univalently onto this disk.

The question about lower estimation of d_h for univalent analytic functions first was considered in papers of Koebe [16] and Bieberbach [2] in connection with the well known problem of covering disk in the class $\mathcal{S}$. Here $\mathcal{S}$ denotes the classical family of all normalized univalent (analytic) functions in $\mathbb{D}$ investigated by a number of researchers (see [12, 14, 21]). In the class of analytic functions h in $\mathbb{D}$ with $h'(0) = 1$, the determination of the exact value of the greatest lower bound of all d_h is one of the most important problems in geometric function theory of one complex variable. For historical discussion of the attempts of various mathematicians to estimate the lower bound for $d_h(z)$, we refer to [18] and also [4, 6, 7] for recent developments.

If $\mathcal{LU}$ denotes the family of functions h analytic and locally univalent ($h'(z) \neq 0$) in $\mathbb{D}$, then the classical Schwarz lemma for analytic functions gives the following well-known sharp upper estimate for $d_h(z)$:

$$d_h(z) \le |h'(z)|(1 - |z|^2).$$

Often the right hand side quantity, namely, $r(h(z), h(\mathbb{D})) = |h'(z)|(1 - |z|^2)$ is referred to as the conformal radius of the domain $h(\mathbb{D})$ at $h(z)$. Sharp and nontrivial lower estimate for $d_h(z)$ was obtained by Pommerenke [20] in a detailed analysis of

2000 *Mathematics Subject Classification.* Primary: 30C62, 31A05; Secondary: 30C45,30C75.

Key words and phrases. Locally univalent harmonic mappings, linear and affine invariant families, convex and close-to-convex functions, and covering theorems.

what is called *linear invariant families* of locally univalent analytic functions in $\mathbb{D}$. Throughout we denote by Aut $(\mathbb{D})$, the set of all conformal automorphisms (Möbius self-mappings) $\phi(z) = e^{i\theta}\frac{z+a}{1+\overline{a}z}$, where $|a| < 1$ and $\theta \in \mathbb{R}$, of the unit disk $\mathbb{D}$.

Definition 1. (cf. [20]) A non-empty collection $\mathfrak{M}$ of functions from $\mathcal{LU}$ is called a linear invariant family (LIF) if for each $h \in \mathfrak{M}$, normalized such that $h(z) = z + \sum_{k=2}^{\infty} a_k(h)z^k$, the functions $H_\phi(z)$ defined by

$$H_\phi(z) = \frac{h(\phi(z)) - h(\phi(0))}{h'(\phi(0))\phi'(0)} = z + \dots,$$

belong to $\mathfrak{M}$ for each $\phi \in \text{Aut}(\mathbb{D})$.

The order of the family $\mathfrak{M}$ is defined to be $\alpha := \texttt{ord}\,\mathfrak{M} = \sup_{h\in\mathfrak{M}} |a_2(h)|$. The universal LIF, denoted by $\mathcal{U}_\alpha$, is defined to be the collection of all linear invariant families $\mathfrak{M}$ with order less than or equal to α (see [20]). An interesting fact about the order of a LIF family is that many properties of it depend only on the order of the family. It is well-known [20] that $\mathcal{U}_\alpha \neq \emptyset$ if and only if $\alpha \geq 1$. The family $\mathcal{U}_1$ is precisely the family $\mathcal{K}$ of all normalized convex univalent (analytic) functions whereas $\mathcal{S} \subset \mathcal{U}_2$.

Note that $\mathcal{U}_\alpha$ is the largest LIF of functions h with the restriction of growth (see [26]):

$$|h'(z)| \leq \frac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}.$$

In [20], Pommerenke has proved that for each $h \in \mathcal{U}_\alpha$ the following sharp lower estimate of $d_h(z)$ holds:

$$d_h(z) \geq \frac{1}{2\alpha}|h'(z)|(1-|z|^2).$$

In the present paper we obtain estimate of the functional $d_f(z)$ when instead of analytic functions $h(z)$ we consider harmonic locally univalent mappings

$$f(z) = h(z) + \overline{g(z)} = \sum_{k=1}^{\infty} \left(a_k z^k + a_{-k}\overline{z}^k\right), \tag{1.1}$$

i.e. when $\overline{g(z)}$ is added to the functions h. In the above decomposition of f, the functions h and g are called the analytic and co-analytic parts of f, respectively. We say that a harmonic functions $f = h + \overline{g}$ is sense-preserving if the Jacobian $J_f(z) = |h'(z)|^2 - |g'(z)|^2$ of f is positive. Lewy's theorem [17] (see also for example [13, Chapter 2, p. 20] and [22]) implies that every harmonic function f on $\mathbb{D}$ is locally one-to-one and sense-preserving on $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$. Note that $J_f(z) > 0$ in $\mathbb{D}$ if and only if $h'(z) \neq 0$ and there exists an analytic function ω_f in $\mathbb{D}$ such that

$$|\omega_f(z)| < 1 \quad \text{for } z \in \mathbb{D}, \tag{1.2}$$

where $\omega_f(z) = g'(z)/h'(z)$. Here ω_f is referred to as the (complex) *dilatation* of the harmonic mapping $f = h + \overline{g}$. When it is convenient, we simply use the notation ω instead of ω_f.

There are different generalizations of the notion of the linear invariant family to the case of harmonic mappings. For example, the question about a lower estimate of the radius $d_f(0)$ of the univalent disk centered at the origin was examined by Sheil-Small [24] in the linear and affine invariant families of univalent harmonic functions f. There are a number of articles in the literature proving such inequalities or studying the related mappings in various settings. For example, see [4, 5, 6, 7, 8, 9, 10, 27, 29], and also the work from [3] in which one can obtain a lower bound on the radius for quasi-regular mappings. The concept of linear and affine invariance was also discussed by Schaubroeck [23] for the case of locally univalent harmonic mappings.

Definition 2. The family $\mathcal{LU}_H$ of locally univalent sense-preserving harmonic functions f in the disk $\mathbb{D}$ of the form (1.1) is called a *linear invariant family* (LIF) if for each $f = h + \overline{g} \in \mathcal{LU}_H$ the following conditions are fulfilled: $a_1 = 1$ and

$$\frac{f(\phi(z)) - f(\phi(0))}{h'(\phi(0))\phi'(0)} \in \mathcal{LU}_H$$

for each $\phi \in \operatorname{Aut}(\mathbb{D})$. A family $\mathcal{AL}_H$ is called *linear and affine invariant* (ALIF) if it is LIF and in addition each $f \in \mathcal{AL}_H$ satisfies the condition that

$$\frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon\overline{f_{\overline{z}}(0)}} \in \mathcal{AL}_H \quad \text{for every} \quad \varepsilon \in \mathbb{D}.$$

The number $\texttt{ord}\,\mathcal{AL}_H = \sup_{f\in\mathcal{AL}_H} |a_2|$ is known as the order of the ALIF $\mathcal{AL}_H$.

The order of LIF $\mathcal{LU}_H$ without the assumption of affine invariance property is defined in the same way: $\texttt{ord}\,\mathcal{LU}_H = \sup_{f\in\mathcal{LU}_H} |a_2|$.

Throughout the discussion, we suppose that the orders of these families, namely, $\texttt{ord}\,\mathcal{AL}_H$ and $\texttt{ord}\,\mathcal{LU}_H$, are finite. The universal linear and affine invariant family, denoted by $\mathcal{AL}_H(\alpha)$, is the largest ALIF $\mathcal{AL}_H$ of order $\alpha = \texttt{ord}\,\mathcal{AL}_H$. Thus, the subfamily $\mathcal{AL}_H^0$ of ALIF $\mathcal{AL}_H$ consists of all functions $f \in \mathcal{AL}_H$ such that $f_{\overline{z}}(0) = 0$. If $f \in \mathcal{AL}_H^0$ is univalent in $\mathbb{D}$, then according to the result of Sheil-Small [24] one has the following sharp lower estimate:

$$d_f(0) \geq \frac{1}{2\alpha}. \tag{1.3}$$

For $\alpha > 0$ and $Q \geq 1$, denote by $\mathcal{H}(\alpha, Q)$ the set of all locally univalent Q-quasiconformal harmonic mappings $f = h + \overline{g}$ in $\mathbb{D}$ of the form (1.1) with the normalization $a_1 + a_{-1} = 1$ such that

$$h(z)/h'(0) \in \mathcal{U}_\alpha, \;\; |g'(z)/h'(z)| \leq k, \;\; k = (Q-1)/(Q+1) \in [0, 1).$$

The family $\mathcal{H}(\alpha, Q)$ was introduced and investigated in details [27, 28]. In particular, he established double-sided estimates of the value $d_f(z)$ for functions belonging to the family $\mathcal{H}(\alpha, Q)$ (see [29]).

Note that the classes $\mathcal{H}(\alpha, Q)$, which expand with the increasing values of $\alpha \in [1, \infty]$ and $Q \in [1, \infty]$, cover all sense-preserving locally quasiconformal harmonic mappings with the indicated normalization.

We shall restrict ourselves to the case of finite Q. In [27, 28], it was also shown that the family $\mathcal{H}(\alpha,Q)$ possess the property of linear invariance in the following sense: for each $f=h+\overline{g}\in\mathcal{H}(\alpha,Q)$ and for every $\phi(z)=e^{i\theta}\frac{z+a}{1+\overline{a}z}\in \operatorname{Aut}(\mathbb{D})$, the transformation

$$\frac{f(\phi(z))-f(\phi(0))}{\partial_\theta f(\phi(0))|\phi'(0)|}\in\mathcal{H}(\alpha,Q), \tag{1.4}$$

where $\partial_\theta f(z)=h'(z)e^{i\theta}+\overline{g'(z)e^{i\theta}}$ denotes the directional derivative of the complex-valued function f in the direction of the unit vector $e^{i\theta}$.

In [29], Starkov proved that for each $f\in\mathcal{H}(\alpha,Q)$ and $z\in\mathbb{D}$,

$$\frac{1-|z|^2}{2\alpha Q}\max_\theta|\partial_\theta f(z)|\le d_f(z)\le Q(1-|z|^2)\min_\theta|\partial_\theta f(z)|$$

which is equivalent to

$$\frac{1-|z|^2}{2\alpha Q}\left(|h'(z)|+|g'(z)|\right)\le d_f(z)\le Q(1-|z|^2)\left(|h'(z)|-|g'(z)|\right), \tag{1.5}$$

and the lower estimate is sharp in contrast to the upper one.

One of the main aims of this article is to establish sharp estimations of the ratio $d_f(z)/d_h(z)$ for Q-quasiconformal harmonic mappings $f=h+\overline{g}$. In particular, sharp upper estimate in (1.5) is obtained. The ratio $d_f(z)/d_h(z)$ demonstrates how the radius of the largest univalent disk with the center at $h(z)$ on the manifold $h(\mathbb{D})$ varies if we add, to the analytic function h, the function $\overline{g}$.

We now state our first result.

Theorem 1. *Let $f=h+\overline{g}\in\mathcal{H}(\alpha,Q)$ for some $Q\in[1,\infty]$, and $\omega(z)=g'(z)/h'(z)$ be the complex dilatation of the mapping f. Then for $z\in\mathbb{D}$,*

$$1-k\le m\left(\frac{|\omega(z)|}{k},Q\right)\le\frac{d_f(z)}{d_h(z)}\le M\left(\frac{|\omega(z)|}{k},k\right)\le 1+k, \tag{1.6}$$

where $k=(Q-1)/(Q+1)\in[0,1]$. Here the functions $M(.,k)$ and $m(.,Q)$ are defined as follows:

$$M(x,k)=\begin{cases}1+\dfrac{k}{x}\left\{1-\left(\dfrac{1}{x}-x\right)\log(1+x)\right\} & \text{when } x\in(0,1]\\ \lim\limits_{x\to0^+}M(x,k)=1+\dfrac{k}{2} & \text{when } x=0\end{cases}, \tag{1.7}$$

and

$$\frac{1}{m(x,Q)}=\begin{cases}\displaystyle\int_0^1\frac{1+\varphi^{-1}(\varphi(t)/Q)x}{1-kx+\varphi^{-1}(\varphi(t)/Q)(x-k)}\,dt & \text{when } Q<\infty\\ 0 & \text{when } Q=\infty\end{cases}, \tag{1.8}$$

with

$$\varphi(t)=\frac{\pi}{2}\frac{\mathcal{K}'(t)}{\mathcal{K}(t)}\quad(t\in(0,1))$$

where $\mathcal{K}$ denotes the (Legendre) complete elliptic integral of the first kind given by

$$\mathcal{K}(t) = \int_0^{\pi/2} \frac{dx}{\sqrt{1 - t^2 \sin^2 x}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-t^2x^2)}}$$

and $\mathcal{K}'(t) = \mathcal{K}(\sqrt{1-t^2})$. The argument t is sometimes called the modulus of the elliptic integral $\mathcal{K}(t)$.

Estimations in (1.6) *are sharp for the family $\mathcal{H}(\alpha, Q)$ for $Q < \infty$ and for each $\alpha \geq 1$. When $Q = \infty$, estimations in* (1.6) *are sharp in the sense that for each $z \in \mathbb{D}$,*

$$\inf_{f \in \mathcal{H}(\alpha,\infty)} \frac{d_f(z)}{d_h(z)} = m(x, \infty) = 0 \quad \textit{and} \quad \sup_{f \in \mathcal{H}(\alpha,\infty)} \frac{d_f(z)}{d_h(z)} = M(1,1) = 2.$$

Remark 1. For fixed $\zeta \in \mathbb{D}$, the least value of the upper estimation in (1.6) is attained when $x = 0$; that is when $\omega(\zeta) = 0$. In this case the estimation in (1.6) takes the form

$$\frac{d_f(\zeta)}{d_h(\zeta)} \leq 1 + \frac{k}{2}.$$

Suppose that $f = h + \overline{g} \in \mathcal{H}(\alpha, Q)$, $\alpha \in [1, \infty]$, and $f_1(z) = C \cdot f(z) = h_1(z) + \overline{g_1(z)}$, where C is a complex constant. Then the following relations hold:

$$d_{f_1}(z) = |C|\, d_f(z) \ \text{ and } \ d_{h_1}(z) = |C|\, d_h(z), \quad z \in \mathbb{D}.$$

Moreover, after appropriate normalization, every Q-quasiconformal harmonic mapping in $\mathbb{D}$ belongs to the family $\mathcal{H}(\alpha, Q)$ for some α. Therefore an equivalent formulation of Theorem 1 may now be stated.

Theorem 2. *Let $f = h + \overline{g}$ be a locally univalent Q-quasiconformal harmonic mapping of the disk $\mathbb{D}$, $Q \in [1, \infty]$, and $\omega(z) = g'(z)/h'(z)$. Then the inequalities* (1.6) *continue to hold and the estimations in* (1.6) *are sharp.*

Next, we consider $f = h + \overline{g} \in \mathcal{H}(\alpha, Q)$ and introduce $H(z) = h(z)/h'(0)$ from $\mathcal{U}_\alpha$. Then we have (see [27, 28])

$$\frac{1}{1+k} \leq |h'(0)| \leq \frac{1}{1-k}$$

and thus,

$$\frac{d_H(z)}{1+k} \leq d_h(z) = |h'(0)| \cdot d_H(z) \leq \frac{d_H(z)}{1-k}.$$

These inequalities and (1.6) give the following.

Corollary 1. *Let $f = h + \overline{g} \in \mathcal{H}(\alpha, Q)$ and $h(z) = h'(0)H(z)$. Then we have*

$$\frac{d_H(z)}{Q} \leq d_f(z) \leq Q\, d_H(z) \quad \textit{for } z \in \mathbb{D}.$$

The sharpness of the last double-sided inequalities at the point $z = 0$ follows from the proof of Theorem 1.

We now state the remaining results of the article.

Theorem 3. *Let $f = h + \overline{g}$ be a locally quasiconformal harmonic mapping belonging to the family $\mathcal{AL}_H$ with $\mathtt{ord}(\mathcal{AL}_H) = \alpha < \infty$, $\omega(z) = g'(z)/h'(z)$ and $|\omega(z)| < 1$. Then*

$$d_f(z) \geq \frac{1 - |\omega(z)|}{2\alpha} \left(\frac{1 - |z|}{1 + |z|}\right)^{\alpha} \quad \text{for } z \in \mathbb{D}. \tag{1.9}$$

The estimation $d_f(0)$ is sharp for example in the universal ALIF $\mathcal{AL}_H(\alpha)$.

Recall that a locally univalent function f is said to be convex in the disk $\mathbb{D}(z_0, r) := \{z : |z - z_0| < r\}$ if f maps $\mathbb{D}(z_0, r)$ univalently onto a convex domain. The radius of convexity of the family $\mathcal{F}$ of functions defined on the disk $\mathbb{D}$ is the largest number r_0 such that every function $f \in \mathcal{F}$ is convex in the disk $\mathbb{D}(0, r_0)$.

Theorem 4. *If $f \in \mathcal{H}(\alpha, Q)$, then for every $z \in \mathbb{D}$, the function f is convex in the disk $\mathbb{D}(z, R(z))$, where*

$$R(z) = \frac{1}{2}\left(R_0 + R_0^{-1} - \sqrt{\left(R_0 - R_0^{-1}\right)^2 + 4|z|^2}\right), \tag{1.10}$$

and

$$R_0 = \alpha + k^{-1} - \sqrt{k^{-2} - 1} - \sqrt{\left(\alpha + k^{-1} - \sqrt{k^{-2} - 1}\right)^2 - 1}. \tag{1.11}$$

In particular, the radius of convexity of the family $\mathcal{H}(\alpha, Q)$ is no less than R_0.

The proofs of Theorems 1, 3 and 4 will be presented in Section 2.

2. Proofs of the Main results

2.1. **Proof of Theorem 1.** The proof of the theorem is divided into three parts.

Part 1: Let $f = h + \overline{g}$ satisfy the assumptions of Theorem 1. In compliance with the definition of the value $d_f(0)$, there exists a boundary point A of the manifold $f(\mathbb{D})$ such that $A \in \{w : |w| = d_f(0)\}$. Consider the smooth curve $\ell_0 = f^{-1}([0, A))$, namely, the preimage of the semi-open segment $[0, A)$ with the starting point 0 in the disk $\mathbb{D}$. Then

$$d_f(0) = |A| = \left|\int_{\ell_0} df(z)\right| = \min_{\gamma}\left|\int_{\gamma} df(z)\right|,$$

where the minimum is taken over all smooth paths $\gamma(t)$, $t \in [0, 1)$, such that $\gamma(0) = 0$, $|\gamma(t)| < 1$ and $\lim_{t \to 1^-} |\gamma(t)| = 1$.

Similarly we define the value

$$d_h(0) = |B| = \left|\int_{\ell} dh(z)\right| = \min_{\gamma}\left|\int_{\gamma} dh(z)\right|,$$

where the simple smooth curve $\ell = h^{-1}([0, B))$ is emerging from the origin, the preimage of the semi-open segment $[0, B)$ under the mapping h. Consider the following parametrization of the curve ℓ: $\ell(t) = h^{-1}(Bt)$, $t \in [0, 1)$. Then $h'(\ell(t))\ell'(t) = B$

and

$$
\begin{aligned}
d_f(0)=\left|\int_0^1 df(\ell_0(t))\right| &\leq \left|\int_0^1 df(\ell(t))\right| \\
&= \left|\int_0^1 \left\{h'(\ell(t))\ell'(t)+\overline{g'(\ell(t))\ell'(t)}\right\} dt\right| \\
&= |B|\left|\int_0^1 \left\{1+\frac{\overline{g'(\ell(t))\ell'(t)}}{h'(\ell(t))\ell'(t)}\right\} dt\right| \\
&\leq d_h(0)\left\{1+\int_0^1 |\omega(\ell(t))|\, dt\right\}. \qquad (2.1)
\end{aligned}
$$

At first we consider the case $k=\sup_{z\in\mathbb{D}}|\omega(z)|<1$. Since $|\omega(z)|\leq k$ for $z\in\mathbb{D}$, we have

$$\omega(0)/k=\overline{a_{-1}}/(k\,a_1)=:u\in\overline{\mathbb{D}}.$$

If $|u|=1$ for $k<1$, then we have the inequality

$$d_f(0)\leq d_h(0)(1+k)=d_h(0)M(1,k)$$

which proves the upper estimate in the inequality (1.6) for $z=0$.

Let us now assume that $|u|<1$ for some $k<1$. Then, from a generalized version of the classical Schwarz lemma (see for example [14, Chapter VIII, §1]), it follows that

$$\frac{|\omega(z)|}{k}\leq\frac{|z|+|u|}{1+|u|\,|z|}. \qquad (2.2)$$

Consequently, by (2.1), one has

$$d_f(0)\leq d_h(0)\left\{1+k\int_0^1\frac{|\ell(t)|+|u|}{1+|u|\,|\ell(t)|}\,dt\right\}. \qquad (2.3)$$

Also, the function $h^{-1}(B\zeta)$ maps biholomorphically $\mathbb{D}$ onto some subdomain of the disk $\mathbb{D}$. Applying the classical Schwarz lemma, we obtain the inequality $|h^{-1}(B\zeta)|\leq|\zeta|$ and hence, $|\ell(t)|\leq t$ holds. Using the last estimate and the inequality (2.3), one can obtain, after evaluating the integral, the inequality

$$d_f(0)\leq d_h(0)\left\{1+k\int_0^1\frac{t+|u|}{1+|u|t}\,dt\right\}=d_h(0)M(|u|,k),$$

where $M(x,k)$ is defined by (1.7). The function $M(x,k)$ is strictly increasing on $(0,1]$ with respect to the variable x and for each fixed $k\in[0,1]$. This follows from the observation that (see (1.7))

$$\frac{\partial M(x,k)}{\partial x} = -\frac{k}{x^2}+\frac{2k}{x^3}\log(1+x)-k\left(\frac{1-x}{x^2}\right),$$

which is positive, since $\log(1+x)>x-x^2/2$. Hence

$$d_f(0)\leq d_h(0)M(|u|,k)\leq d_h(0)M(1,k)=(1+k)d_h(0). \qquad (2.4)$$

We now set $k=1$. According to Lewy's theorem [17] for locally univalent harmonic mapping f, we obtain that $|\omega(z)|\neq 1$ for all $z\in\mathbb{D}$. Next we obtain the inequality (2.4) in the case $k=1$ by repeating the argument of the case $k<1$.

We now begin to prove that the upper estimate in (1.6) is true for all $\zeta\in\mathbb{D}$. As mentioned above, the family $\mathcal{H}(\alpha,Q)$ is linear invariant in the sense of [27, 28] (see (1.4) above). Hence, for each fixed $\zeta=re^{i\theta}\in\mathbb{D}$ ($r\in[0,1)$, $\theta\in\mathbb{R}$), the function F defined by

$$F(z)=\frac{f\left(e^{i\theta}\frac{z+r}{1+rz}\right)-f(re^{i\theta})}{\partial_\theta f(re^{i\theta})(1-r^2)}=H(z)+\overline{G(z)}$$

belongs to the family $\mathcal{H}(\alpha,Q)$, where H and G are analytic in $\mathbb{D}$ such that $H(0)=G(0)=0$. Therefore, in view of (2.4) for $k\in[0,1]$, we have

$$d_F(0)=\frac{d_f(\zeta)}{|\partial_\theta f(\zeta)|(1-|\zeta|^2)}\leq d_H(0)M(x,k),$$

where $x=|G'(0)/H'(0)|/k=|\omega(\zeta)|/k\in[0,1]$ if $k\in[0,1)$, and $x=|G'(0)/H'(0)|=|\omega(\zeta)|\in[0,1)$ when $k=1$. Note that

$$H(z)=\frac{h\left(e^{i\theta}\frac{z+r}{1+rz}\right)-h(re^{i\theta})}{\partial_\theta f(re^{i\theta})(1-r^2)}.$$

Consequently,

$$d_H(0)=\frac{d_h(\zeta)}{|\partial_\theta f(\zeta)|(1-|\zeta|^2)}$$

so that

$$d_f(\zeta)\leq d_h(\zeta)M(x,k)\leq(1+k)d_h(\zeta)$$

and we complete the proof of the upper estimate in (1.6).

Part 2: We now deal with the sharpness of the upper estimate in (1.6). Consider the case $k\in[0,1)$. For every $\alpha\in\mathbb{N}$ and every $\zeta\in\mathbb{D}$, we shall indicate functions from the families $\mathcal{H}(\alpha,Q)$ such that $d_f(\zeta)/d_h(\zeta)=M(x)=1+k$, where $x=|\omega(\zeta)|/k$. Since the families $\mathcal{H}(\alpha,Q)$ are enlarging with increasing values of α, the sharpness of the upper estimate in (1.6) will be shown for every $\zeta\in\mathbb{D}$ and each $\alpha\in[1,\infty]$.

Consider the sequence $\{k_n\}_{n=1}^\infty$ of functions from $\mathcal{U}_n$ defined by

$$k_n(z)=\frac{i}{2n}\left[\left(\frac{1-iz}{1+iz}\right)^n-1\right].$$

Then we have $d_{k_n}(0)=1/2n$ (see [20]) and observe that k_n maps the unit disk $\mathbb{D}$ univalently onto the Riemann surface $k_n(\mathbb{D})$ whose boundary described by

$$\partial k_n(\mathbb{D})=\left\{\frac{i}{2n}[(i\lambda)^n-1]:\ \lambda\in\mathbb{R}\right\}=\left\{\frac{i}{2n}[s\,e^{\pm i\pi n/2}-1]:\ s\geq 0\right\}$$

consists of two rays. Then the univalent image of the disk $\mathbb{D}$ under the mapping

$$f_n(z)=h_n(z)+\overline{g_n(z)}=\frac{1}{1-k}[k_n(z)-k\,\overline{k_n(z)}]\in\mathcal{H}(n,Q),\quad k\in[0,1),$$

($a_1 = \frac{1}{1-k}$, $a_{-1} = -\frac{k}{1-k}$) represents the manifold with the boundary

$$\partial f_n(\mathbb{D}) = \left\{ \frac{i}{2n(1-k)}[s(e^{\pm i\pi n/2} + k\, e^{\mp i\pi n/2}) - 1 - k] : \, s \geq 0 \right\},$$

which consists of two rays parallel to the coordinate axes and arising from the point $-\frac{i}{2n}Q$. Note that the function f_n maps the semi-open segment $[0, -i)$ bijectively onto $[0, -\frac{i}{2n}Q)$ and thus, we conclude that

$$d_{f_n}(0) = \frac{Q}{2n}.$$

This gives

$$d_{f_n}(0) = d_{k_n}(0)Q = d_{h_n}(0)(1+k),$$

where $h_n(z) = k_n(z)/(1-k)$. The sharpness of the upper estimate in (1.6) is proved for $\zeta = 0$ and $k < 1$.

Next we let $0 \neq \zeta \in \mathbb{D}$, $k < 1$, and consider a conformal automorphism $\phi(z) = (z+\zeta)/(1+\overline{\zeta}z)$ of the unit disk $\mathbb{D}$. Then the inverse mapping is given by $\phi^{-1}(z) = (z-\zeta)/(1-\overline{\zeta}z)$. From the condition (1.4) of the linear invariance property of the family $\mathcal{H}(\alpha, Q)$, it follows that the function f defined by

$$f(z) = \frac{f_n(\phi^{-1}(z)) - f_n(-\zeta)}{\partial_0 f_n(-\zeta)(1-|\zeta|^2)} = h(z) + \overline{g(z)}$$

belongs to $\mathcal{H}(\alpha, Q)$, where h and g have the same meaning as above. Taking into account of the normalization condition for functions in the family $\mathcal{H}(\alpha, Q)$, we deduce that

$$\frac{f(\phi(z)) - f(\zeta)}{\partial_0 f(\zeta)(1-|\zeta|^2)} = f_n(z) = h_n(z) + \overline{g_n(z)}.$$

Therefore,

$$d_{f_n}(0) = \frac{d_f(\zeta)}{|\partial f_0(\zeta)|(1-|\zeta|^2)} = d_{h_n}(0)(1+k).$$

On the other hand, a direct computation gives

$$h_n(z) = \frac{h(\phi(z)) - h(\zeta)}{\partial_0 f(\zeta)(1-|\zeta|^2)} \quad \text{and} \quad d_{h_n}(0) = \frac{d_h(\zeta)}{|\partial f_0(\zeta)|(1-|\zeta|^2)}$$

showing that

$$d_{f_n}(0)|\partial f_0(\zeta)|(1-|\zeta|^2) = d_f(\zeta) = d_h(\zeta)(1+k),$$

which completes the proof of the upper estimation in Theorem 1 for $k \in [0, 1)$.

If $k = 1$ then for $j \in \mathbb{N}$, we consider the sequence $\{f_{n,j}\}$ of functions

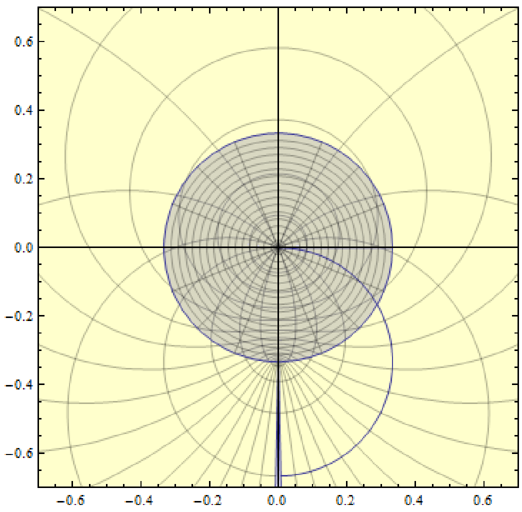
$$f_{n,j}(z) = h_{n,j}(z) + \overline{g_{n,j}(z)} = jk_n(z) - (j-1)\,\overline{k_n(z)}.$$

We see that $f_{n,j} \in \mathcal{H}(n, 2j-1) \subset \mathcal{H}(n, \infty)$ for each $j \in \mathbb{N}$. Therefore,

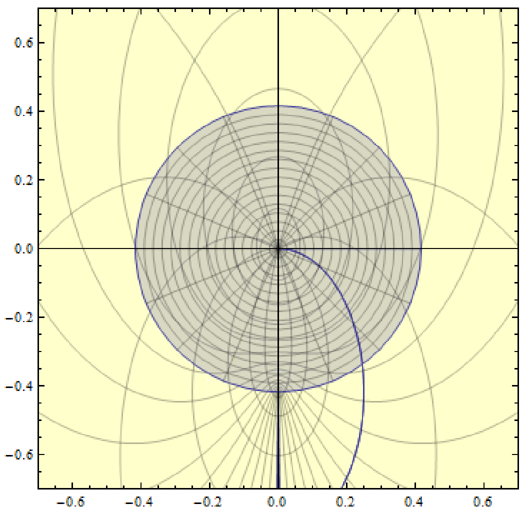
$$d_{f_{n,j}}(0) = d_{h_{n,j}}(0)M(1, 1-1/j).$$

Hence

$$\sup_{j \in \mathbb{N}} \frac{d_{f_{n,j}}(0)}{d_{h_{n,j}}(0)} = M(1,1) = 2.$$



(a) h_2



(b) f_2

FIGURE 1. The covering disks for functions h_2 and f_2 for $k = 0.25$ with centers at origin and radii $1/3$ and $5/12$, respectively.

The sharpness of the upper estimation in (1.6) for $k = 1$, $\zeta \neq 0$, can be proved analogously. So, we omit the details.

The images of polar grid in the unit disk under mappings h_2 and f_2 are indicated in Figures 1(a)-(b) which illustrate the sharpness assertions proved in the above estimations.

Part 3: Finally, we deal with lower estimation of $d_f(z)$. If $k = 1$, then the lower estimation in (1.6) is trivial because $m(x, \infty) = 0$. So, we may assume that $k \in [0, 1)$. As in Part 1, we define the boundary points A and B of the manifolds $f(\mathbb{D})$ and $h(\mathbb{D})$, respectively, and smooth curves $\ell_0 = f^{-1}([0, A))$ and ℓ in the same manner as

in Part 1. Consider the parametrization of the curve ℓ_0:

$$\ell_0(t) = f^{-1}(At),\ t \in [0,1).$$

Then $df(\ell_0(t)) = Adt$ and thus,

$$\begin{aligned}
d_h(0) &= \left|\int_0^1 dh(\ell(t))\right| \\
&\leq \left|\int_0^1 h'(\ell_0(t))\ell_0'(t)\,dt\right| \\
&= \left|\int_0^1 \left\{h'(\ell_0(t))\ell_0'(t) + \overline{g'(\ell_0(t))\ell_0'(t)}\right\}\right. \\
&\quad \left.\times \left(1 - \frac{\overline{g'(\ell_0(t))\ell_0'(t)}}{h'(\ell_0(t))\ell_0'(t) + \overline{g'(\ell_0(t))\ell_0'(t)}}\right) dt\right| \\
&= \left|\int_0^1 \frac{h'(\ell_0(t))\ell_0'(t)}{h'(\ell_0(t))\ell_0'(t) + \overline{g'(\ell_0(t))\ell_0'(t)}} df(\ell_0(t))\right| \\
&\leq |A| \int_0^1 \frac{dt}{1 - |\omega(\ell_0(t))|}. \qquad (2.5)
\end{aligned}$$

In view of the inequality (2.2), we find that

$$|\omega(\ell_0(t))| \leq k\frac{|\ell_0(t)| + x}{1 + x|\ell_0(t)|}, \tag{2.6}$$

where $x = |\omega(0)|/k$.

It is possible to obtain an estimate for $|\ell_0(t)| = |f^{-1}(At)|$, $t \in [0,1)$, with the help of the analog of the Schwarz lemma for Q-quasiconformal automorphisms of the disk. Let F be a Q-quasiconformal automorphism of $\mathbb{D}$, and $F(0) = 0$. It is known (see for example [1, Chapter 10, equality (10.1)]) that the sharp estimation

$$|F(z)| \leq \varphi^{-1}\left(Q^{-1}\varphi(|z|)\right)$$

holds, where φ and Q are as in the statement. The function $f^{-1}(Aw)$ defined on the unit disk $\{w : |w| < 1\}$ satisfies the conditions $f^{-1}(0) = 0$ and $|f^{-1}(Aw)| < 1$. Let Φ be the univalent conformal mapping of the domain $f^{-1}(A\mathbb{D})$ onto the unit disk $\mathbb{D}$ and $\Phi(0) = 0$. Then the composition $\Phi \circ f^{-1}(Az)$ is a Q-quasiconformal automorphism of $\mathbb{D}$ and Φ^{-1} satisfies the conditions of the classical Schwarz lemma for analytic functions. Hence, we have

$$|\ell_0(t)| = |\Phi^{-1}(\Phi \circ f^{-1}(At))| \leq |\Phi \circ f^{-1}(At)| \leq \varphi^{-1}(Q^{-1}\varphi(t)).$$

As a result of it and taking into account of the last estimation, inequalities (2.5) and (2.6), and the fact that the function $(1 + y\,x)/(1 - kx + y(x - k))$ is strictly increasing with respect to y on $(0,1)$, we conclude that

$$d_h(0) \leq d_f(0) \int_0^1 \frac{1 + y\,x}{1 - kx + y(x - k)}\,dt \leq \frac{d_f(0)}{1 - k},$$

where $y = \varphi^{-1}(Q^{-1}\varphi(t)) \leq 1$ for $t \in (0,1)$. Therefore the lower estimate in (1.6) is sharp at the origin.

The proof of the lower estimation in (1.6) for $0 \neq \zeta \in \mathbb{D}$ follows easily if we proceed with the same manner as in Part 1 and use the linear invariance property of the family $\mathcal{H}(\alpha, Q)$.

For the sharpness of the left side of the inequality in (1.6) for $k \in [0,1)$, we consider the functions (see [27, 28])

$$h_\alpha(z) = \frac{1}{2i\alpha}\left[\left(\frac{1+iz}{1-iz}\right)^\alpha - 1\right] \in \mathcal{U}_\alpha \tag{2.7}$$

and

$$f(z) = h(z) + \overline{g(z)} := \frac{h_\alpha(z)}{1+k} + \frac{k\overline{h_\alpha(z)}}{1+k}.$$

Then it is a simple exercise to see that

$$d_f(0) = \frac{1}{2\alpha Q} \quad \text{and} \quad d_h(0) = \frac{1}{2\alpha(1+k)}.$$

Comparison of radii $d_h(0)$ and $d_f(0)$ and sharpness of the lower estimation of $d_f(0)/d_h(0)$ is illustrated in Figures 2(a)–(b). In these figures, the images of polar grid in the unit disk under mappings $h_\alpha/(1+k)$ and f are indicated.

If $k \to 1^-$ then from the last equality we obtain

$$\lim_{k\to 1^-} d_f(0) = 0 \quad \text{and} \quad \lim_{k\to 1^-} d_h(0) = \frac{1}{4\alpha},$$

so that

$$\inf_{f\in\mathcal{H}(\alpha,\infty)} \frac{d_f(0)}{d_h(0)} = 0.$$

Thus the last equality is sharp not only at the origin but also at points $z \in \mathbb{D}$, in view of the degeneration of functions

$$f(z) = \frac{h_\alpha(z) + k\overline{h_\alpha(z)}}{1+k}$$

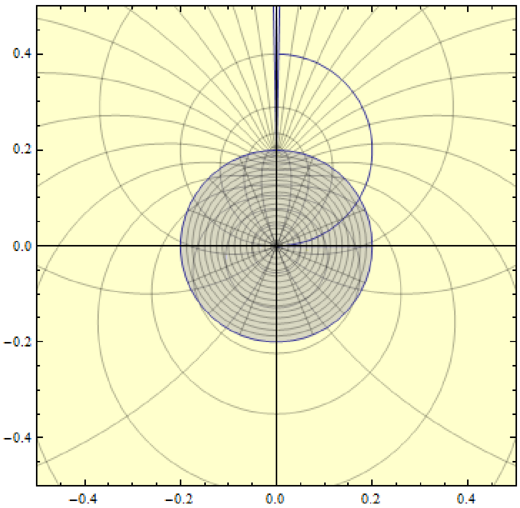
when $k \to 1^-$. The proof of the theorem is complete. □

Remark 2. In some neighbourhood of the origin, it is also possible to obtain a simple lower estimate in the inequality (1.6) without the involvement of elliptic integrals. For example, the well-known theorem of Mori [19] reveals that for Q-quasiconformal automorphism F of the disk $\mathbb{D}$ such that $F(0) = 0$, one has

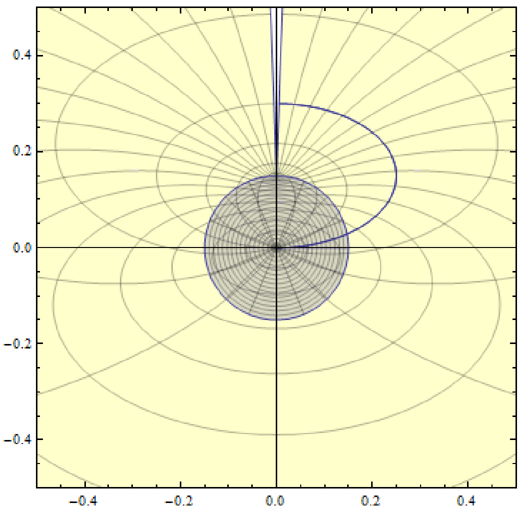
$$|F(z)| \leq 16|z|^{1/Q}.$$

Using this result in the estimation of $|\ell_0(t)|$ in Part 3 of the proof of Theorem 1, one can easily obtain that

$$|\ell_0(t)| = |f^{-1}(At)| = \begin{cases} 16t^{1/Q} \text{ when } 0 \leq t < 1/16^{-Q} \\ \quad 1 \text{ when } 16^{-Q} \leq t < 1. \end{cases}$$



(a) $h_\alpha/(1+k)$



(b) $f = (h_\alpha + k\overline{h}_\alpha)/(1+k)$

FIGURE 2. Covering disks for functions $h_\alpha/(1+k)$ and $f = (h_\alpha + k\overline{h}_\alpha)/(1+k)$ for $k = 0.25$ with centers at origin and radii $1/5$ and $3/20$, respectively.

The last relation provides an opportunity to estimate the ratio $d_h(z)/d_f(z)$ by means of an integral of an elementary function, namely,

$$\begin{aligned}
\frac{d_h(z)}{d_f(z)} &\leq \frac{1}{m(x,Q)} \\
&\leq \frac{1}{1-k}\left(1 - 16^{-Q} + (1-k)\int_0^{16^{-Q}} \frac{1+y\,x}{1-kx+y(x-k)}\,dt\right) \\
&\leq \frac{1}{1-k},
\end{aligned}$$

where $y = 16t^{1/Q} \leq 1$ for $t \in [0, 16^{-Q}]$. Here $x = |\omega(z)|/k,\ z \in \mathbb{D}$.

2.2. **Proof of Theorem 3.** We first prove the inequality (1.9) for $z = 0$. As in the proof of Theorem 1, consider on the circle $\{w : |w| = d_f(0)\}$ the boundary point A of the manifold $f(\mathbb{D})$ and define a curve $\ell_0 = f^{-1}([0, A))$ with the starting point 0 in $\mathbb{D}$. Then

$$d_f(0) = |A| = \left|\int_{\ell_0} df(\zeta)\right| = \int_{\ell_0} |df(\zeta)| \geq \int_{\ell_0} (|h'(\zeta)| - |g'(\zeta)|)\, |d\zeta|. \tag{2.8}$$

In view of the affine invariance property of the family $\mathcal{AL}_H$, the function F defined by

$$F(\zeta) = H(\zeta) + \overline{G(\zeta)} = \frac{f(\zeta) - \varepsilon\overline{f(\zeta)}}{1 - \varepsilon\overline{f_{\overline{z}}(0)}}$$

belongs $\mathcal{AL}_H$ for every ε with $|\varepsilon| < 1$.

For a fixed ζ, we introduce $\theta(z) = \arg h'(z) - \arg g'(z)$ when $g'(z) \neq 0$, and $\theta(z) = \arg h'(z)$ otherwise. Consider then $\varepsilon = se^{i\theta(z)}$ for $s \in [0, 1)$. Therefore, taking into account of the relation $\overline{f_{\overline{z}}(0)} = \omega(0)$, we obtain that

$$H'(\zeta) = \frac{h'(\zeta) - sg'(\zeta)e^{i\theta(z)}}{1 - \varepsilon\omega(0)}$$

and thus,

$$|H'(\zeta)| \leq \frac{|h'(\zeta)| - s|g'(\zeta)|}{1 - s|\omega(0)|}. \tag{2.9}$$

For the other side of the inequality for functions in the family $\mathcal{AL}_H$, the inequality

$$|H'(\zeta)| \geq \frac{(1 - |\zeta|)^{\alpha-1}}{(1 + |\zeta|)^{\alpha+1}} \tag{2.10}$$

holds, where $\alpha = \texttt{ord}\,(\mathcal{AL}_H)$ is defined as in the sense of Definition 2. The inequality (2.10) was obtained in [24] for ALIF of univalent harmonic mappings, but the proof is still valid without a change for any ALIF $\mathcal{AL}_H$ of finite order α. Using inequalities (2.9) and (2.10), we obtain the inequality

$$|h'(\zeta)| - s|g'(\zeta)| \geq (1 - s|\omega(0)|)\frac{(1 - |\zeta|)^{\alpha-1}}{(1 + |\zeta|)^{\alpha+1}}$$

for every $s \in (0, 1)$. Allowing in the last inequality $s \to 1^-$ and substituting the resulting estimate into (2.8), we easily obtain that

$$\begin{aligned} d_f(0) &\geq (1 - |\omega(0)|)\int_{\ell_0} \frac{(1 - |\zeta|)^{\alpha-1}}{(1 + |\zeta|)^{\alpha+1}}|d\zeta| \\ &\geq (1 - |\omega(0)|)\int_0^1 \frac{(1 - t)^{\alpha-1}}{(1 + t)^{\alpha+1}}\, dt = \frac{1 - |\omega(0)|}{2\alpha}. \end{aligned}$$

If $0 < |z| < 1$, then as in the proof of Theorem 1, we may use the linear invariance property of the family $\mathcal{AL}_H$ in accordance with the function $F_1 \in \mathcal{AL}_H$, where

$$F_1(\zeta) = \frac{f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right) - f(z)}{h'(z)(1-|z|^2)}.$$

In this way, applying the estimation of $d_f(0)$ to the function F_1, we see that

$$d_{F_1}(0) \geq \frac{1-|\omega_f(0)|}{2\alpha}.$$

Also, we have

$$d_{F_1}(0) = \frac{d_f(z)}{|h'(z)|(1-|z|^2)}.$$

It remains to note that $|\omega_{F_1}(0)| = |\omega_f(z)|$ and apply the inequality (2.10) to the function $h'(z)$.

In order to prove the sharpness of the estimate of $d_f(0)$, we first note that the functions $p(z) = h_\alpha(z) + k\overline{h_\alpha(z)}$, where each h_α has the form (2.7), belong to $\mathcal{AL}_H(\alpha)$ for every $k = |\omega(0)| \in [0,1)$. Indeed, for each α, the function p is locally univalent and meet the normalization condition of the family $\mathcal{AL}_H(\alpha)$, and $|p_{zz}(0)/2| = |h''_\alpha(0)/2| = \alpha$. Affiliation of the functions

$$q(z) = \frac{p(\phi(z)) - p(\phi(0))}{h'_\alpha(\phi(0))\phi'(0)} = \frac{h_\alpha(\phi(z)) - h_\alpha(\phi(0))}{h'_\alpha(\phi(0))\phi'(0)} + k\overline{\frac{h_\alpha(\phi(z)) - h_\alpha(\phi(0))}{h'_\alpha(\phi(0))\phi'(0)}},$$

and

$$\begin{aligned} w(z) &= \frac{q(z) + \varepsilon\overline{q(z)}}{1+\varepsilon\overline{q_{\bar{z}}(0)}} \\ &= \frac{h_\alpha(\phi(z)) - h_\alpha(\phi(0))}{h'_\alpha(\phi(0))\phi'(0)} \\ &\quad + \overline{\frac{h_\alpha(\phi(z)) - h_\alpha(\phi(0))}{h'_\alpha(\phi(0))\phi'(0)}} \left(\frac{k + \varepsilon h'_\alpha(\phi(0))\phi'(0)/\overline{(h'_\alpha(\phi(0))\phi'(0))}}{\overline{1+\varepsilon k\, h'_\alpha(\phi(0))\phi'(0)/\overline{(h'_\alpha(\phi(0))\phi'(0))}}}\right) \end{aligned}$$

to the family $\mathcal{AL}_H(\alpha)$ for every conformal automorphism ϕ of the disk $\mathbb{D}$ and every $\varepsilon \in \mathbb{D}$, follow from the membership of the function h_α to the universal LIF $\mathcal{U}_\alpha$. The analogous reasoning is true after the change of order of the linear and affine transforms of the function p.

Therefore, $p = h_\alpha + k\overline{h_\alpha} \in \mathcal{AL}_H(\alpha)$ for each $k \in [0,1)$ and at the same time

$$d_p(0) = \frac{1-|\omega(0)|}{2\alpha},$$

which proves the sharpness of the established estimate in the universal ALIF $\mathcal{AL}_H(\alpha)$. The proof of the theorem is complete. □

Remark 3. Recall that a domain $D \subset \mathbb{C}$ is called close-to-convex if its complement $\mathbb{C} \setminus D$ can be written as an union of disjoint rays or lines. The family $\mathcal{C}_H$ of all univalent sense-preserving harmonic mappings f of the form (1.1) such that $a_1 = 1$ and $f(\mathbb{D})$ is close-to-convex, is ALIF (cf. [24]). Also, the inequality in Theorem 3

is sharp in the ALIF $\mathcal{C}_H$. The order of the family $\mathcal{C}_H$ is proved to be 3 ([11]). The harmonic analog of the analytic Koebe function $k(z) = z/(1-z)^2$ (see for example [13, Chapter 5, p. 82]) is given by

$$F(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3} + \overline{\left(\frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}\right)},$$

where $F \in \mathcal{C}_H$ and $F(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/6]$ which is indeed a domain starlike with respect to the origin. From the affine invariance property of the family $\mathcal{C}_H$, we deduce that for every $b \in [0,1)$, the affine mapping

$$f(z) = F(z) - b\,\overline{F(z)}$$

belongs to $\mathcal{C}_H$ such that $\omega(0) = f_{\overline{z}}(0)/f_z(0) = -b$. The function f is a composition of the univalent harmonic mapping F of the disk $\mathbb{D}$ onto $\mathbb{C} \setminus (-\infty, -1/6]$ and affine transformation $\psi(w) = w - b\overline{w}$. The plane $\mathbb{C}$ with a slit $(-\infty, -1/6]$ under the transformation ψ is the plane with a slit along the ray emanating from the point $\psi(-1/6) = -(1-b)/6$ through the point $\psi(-1) = b - 1 < (b-1)/6$, since $b \in [0,1)$. Therefore, $f(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -(1-b)/6]$ and thus, $d_f(0) = (1-b)/6$ and the lower estimate of $d_f(0)$ is sharp in the ALIF $\mathcal{C}_H$.

In the first part of the present paper, we concerned with the question about the covering of the manifold $f(\mathbb{D})$ by disks. Now we turn our attention on the problem related with the covering of $f(\mathbb{D})$ by convex domains.

Sheil-Small [24] proved that the radius of convexity of the univalent subfamily of the linear and affine invariant family $\mathcal{AL}_H$ of harmonic mappings is equal to

$$r_0 = \alpha - \sqrt{\alpha^2 - 1}, \tag{2.11}$$

where $\alpha = \mathtt{ord}\,(\mathcal{AL}_H)$. Later this result was generalized to the families of locally univalent harmonic mappings [15]. Now we will show the radius of convexity will be altered under the assumption of Q-quasiconformality of functions f.

Lemma 1. *Let $\mathcal{LU}_H(\alpha, Q)$ denote the* LIF *of locally univalent Q-quasiconformal harmonic mappings of the order $\alpha < \infty$, where $Q \le \infty$. Then the affine hull*

$$\mathcal{AL}_H = \left\{ F(z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon\overline{a_{-1}}} \, : \, f \in \mathcal{LU}_H(\alpha, Q), \ \varepsilon \in \mathbb{D} \right\}$$

of the family $\mathcal{LU}_H(\alpha, Q)$ is linear and affine invariant of order no greater than $\alpha + \frac{1-\sqrt{1-k^2}}{k}$, where $k = (Q-1)/(Q+1)$.

Proof. In [25], it was shown that the affine hull of the linear invariant in the sense of Definition 2 of the family of the locally univalent harmonic mappings is the ALIF $\mathcal{AL}_H$. Thus, it remains to determine the estimate of the order of the family $\mathcal{AL}_H$.

We begin with $F = H + \overline{G} \in \mathcal{AL}_H$. Then there exists an $f = h + \overline{g} \in \mathcal{LU}_H(\alpha, Q)$ of the form (1.1) with the additional normalization $f_z(0) = h'(0) = a_1 = 1$, and $\varepsilon \in \mathbb{D}$ such that

$$F(z) = \frac{f(z) + \varepsilon\overline{f(z)}}{1 + \varepsilon g'(0)} = H(z) + \overline{G(z)}.$$

It is easy to compute that

$$A_2 = \frac{H''(0)}{2} = \frac{a_2 + \varepsilon\overline{a}_{-2}}{1 + \varepsilon g'(0)},$$

where $a_2 = h''(0)/2$ and $a_{-2} = \overline{g''(0)}/2$. Taking into account of the relation $g'(z) = \omega(z)h'(z)$, where ω is the complex dilatation of f with $|\omega(z)| < k$, we see that

$$g'(0) = \omega(0) \quad \text{and} \quad g''(0) = h''(0)\omega(0) + h'(0)\omega'(0),$$

so that

$$\overline{a_{-2}} = a_2\omega(0) + \omega'(0)/2.$$

If we apply the Schwarz-Pick lemma (see for example [14, Chapter VIII, §1]) to the function $\omega(z)/k$, then the inequality (1.3) in this case leads to

$$\frac{|\omega'(0)|}{k} \le 1 - \frac{|\omega(0)|^2}{k^2}.$$

Using the expression for a_2, we deduce that

$$\begin{aligned} |A_2| &= \left|\frac{a_2(1 + \varepsilon\omega(0)) + \varepsilon\omega'(0)/2}{1 + \varepsilon\omega(0)}\right| \\ &\le |a_2| + \frac{k}{2}\frac{1 - |\omega(0)/k|^2}{1 - |\omega(0)|} \\ &= |a_2| + \frac{k^2 - |\omega(0)|^2}{2k(1 - |\omega(0)|)} \end{aligned}$$

(since $|\varepsilon| < 1$). Calculating the maximum of the function $u(t) = (k^2 - t^2)/(1 - t)$ over the interval $[0, k]$, we obtain the estimate

$$|A_2| \le |a_2| + \frac{1 - \sqrt{1 - k^2}}{k} \le \alpha + \frac{1 - \sqrt{1 - k^2}}{k} < \alpha + 1.$$

The proof of the lemma is complete. □

Using Lemma 1 and the equality (2.11), one obtains the estimate of the radius of convexity of functions in the family $\mathcal{H}(\alpha, Q)$.

2.3. **Proof of Theorem 4.** Let $f_0 = h_0 + \overline{g_0} \in \mathcal{H}(\alpha, Q)$. It is easy to see that the function f_0 is convex in the same disks as the normalized function

$$f(z) = f_0(z)/h_0'(0) = h(z) + \overline{g(z)}$$

that belongs to some LIF $\mathcal{LU}_H(\alpha, Q)$. So it is enough to prove the theorem for such functions f. We first show that the function f is convex in the disk centered at the origin with radius R_0 defined by (1.11).

Clearly, the function f belongs to the affine hull $\mathcal{AL}_H$ of the family $\mathcal{LU}_H(\alpha, Q)$. In view of Lemma 1, the family $\mathcal{AL}_H$ has the order $\alpha_1 \le \alpha + \frac{1-\sqrt{1-k^2}}{k}$. Taking into consideration of the equality (2.11), we conclude that the function f is convex in the disk of radius $R_0 = \alpha_1 - \sqrt{{\alpha_1}^2 - 1}$ centered at the origin.

We now let $0 \neq z_0 \in \mathbb{D}$. Consider a conformal automorphism Φ of the unit disk $\mathbb{D}$ given by

$$\Phi(\zeta) = e^{i \arg z_0} \left(\frac{\zeta + |z_0|}{1 + |z_0|\zeta} \right).$$

We see that Φ maps the disk $\mathbb{D}(0, R_0)$ onto the disk $\mathbb{D}(z_0, R(z_0))$, where $R(z_0)$ is defined in (1.10). In view of the linear invariance property of the family $\mathcal{LU}_H(\alpha, Q)$, the function F defined by

$$F(\zeta) = \frac{f(\Phi(\zeta)) - f(z_0)}{h'(z_0))\Phi'(0)}$$

belongs to $\mathcal{LU}_H(\alpha, Q)$ and as remarked above, the function F maps the disk $\mathbb{D}(0, R_0)$ onto a convex domain. Therefore, the function

$$f(z) = F(\Phi^{-1}(z)) \cdot h'(z_0))\Phi'(0) + f(z_0)$$

is convex and univalent in the disk $\mathbb{D}(z_0, R(z_0))$. The proof of the theorem is complete. $\square$

Acknowledgements. The authors thank the referee for useful comments. The research was supported by the project RUS/RFBR/P-163 under Department of Science & Technology (India) and the Russian Foundation for Basic Research (project 14-01-92692). The second author is currently on leave from Indian Institute of Technology Madras, India. The third author is also supported by Russian Foundation for Basic Research (project 14-01-00510) and the Strategic Development Program of Petrozavodsk State University.

References

1. G. D. Anderson, M. K. Vamanamurthy, and M. Vuorinen, *Conformal invariants, inequalities, and quasiconformal maps*, John Wiley & Sons, 1997.
2. L. Bieberbach, Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln, *Sitzungsber. Preuss. Akad. Wiss.* **38**(1916), 940–955.
3. H. Chen, P. M. Gauthier and W. Hengartner, Bloch constants for planar harmonic mappings, *Proc. Amer. Math. Soc.* **128**(2000), 3231–3240.
4. Sh. Chen, S. Ponnusamy and A. Rasila, Coefficient estimates, Landau's theorem and Lipschitz-type spaces on planar harmonic mappings, *J. Aust. Math. Soc.* **96**(2014), 198–215.
5. Sh. Chen, S. Ponnusamy and X. Wang, Bloch and Landau's theorems for planar p-harmonic mappings, *J. Math. Anal. Appl.* **373**(2011), 102–110.
6. Sh. Chen, S. Ponnusamy and X. Wang, Landau's theorem and Marden constant for harmonic ν-Bloch mappings, *Bull. Aust. Math. Soc.* **84**(2011), 19–32.
7. Sh. Chen, S. Ponnusamy and X. Wang, Properties of some classes of planar harmonic and planar biharmonic mappings, *Complex Anal. Oper. Theory,* **5**(2011), 901–916.
8. Sh. Chen, S. Ponnusamy and X. Wang, Coefficient estimates and Landau-Bloch's theorem for planar harmonic mappings, *Bull. Malaysian Math. Sciences Soc.* **34**(2)(2011), 255–265.
9. Sh. Chen, S. Ponnusamy and X. Wang, Covering and distortion theorems for planar harmonic univalent mappings, *Arch. Math. (Basel)*, **101**(2013), 285–291.
10. Sh. Chen, S. Ponnusamy and X. Wang, Harmonic mappings in Bergman spaces, *Monatsh. Math.* **170**(2013), 325–342.
11. J. G. Clunie and T. Sheil-Small, Harmonic univalent functions, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **9**(1984), 3–25.

12. P. L. Duren, *Univalent functions* (Grundlehren der mathematischen Wissenschaften 259, New York, Berlin, Heidelberg, Tokyo), Springer-Verlag, 1983.
13. P. Duren, *Harmonic mappings in the plane*, Cambridge Tracts in Mathematics, **156**, Cambridge Univ. Press, Cambridge, 2004.
14. G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, Vol. 26, American Mathematical Society, Providence, R.I. 1969 vi+676 pp.
15. S. Yu. Graf and O. R. Eyelangoli, Differential inequalities in linear and affine-invariant families of harmonic mappings, *Russian Math. (Iz. VUZ)* **54**(10)(2010), 60–62.
16. P. Koebe, Über die Uniformisierung beliebiger analytischer Kurven II, *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Math.-Phys. Kl.* (1907), 633–669.
17. H. Lewy, On the non-vanishing of the Jacobian in certain one-to-one mappings, *Bull. Amer. Math. Soc.* **42**(1936), 689–692.
18. D. Minda, The Bloch and Marden constants, Computational methods and function theory (Valparaso, 1989), 131–142, Lecture Notes in Math., 1435, Springer, Berlin, 1990.
19. A. Mori, On an absolute constant in the theory of quasi-conformal mappings, *J. Math. Soc. Japan.* **8**(1956), 156–166.
20. Ch. Pommerenke, Linear-invariante familien analytischer funktionen. I, *Math. Ann.* **155**(1964), 108–154.
21. Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Göttingen, 1975.
22. S. Ponnusamy and A. Rasila, Planar harmonic and quasiregular mappings, Topics in Modern Function Theory (Editors St.Ruschewyeh and S. Ponnusamy): Chapter in CMFT, *RMS-Lecture Notes Series No.* **19**, 2013, pp. 267–333.
23. L. E. Schaubroeck, Subordination of planar harmonic functions, *Complex Variables* **41**(2000), 163–178.
24. T. Sheil-Small, Constants for planar harmonic mappings, *J. London Math. Soc.* **42**(1990), 237–248.
25. M. Sobczak-Knec, V. V. Starkov, and J. Szynal, Old and new order of linear invariant family of harmonic mappings and the bound for Jacobian, *Annales Universitatis Mariae Curie-Skłodowska. Sectio A. Mathematica,* **V. LXV**(2)(2011), 191–202.
26. V.V. Starkov, Equivalent definitions of linear invariant families, *Materialy XI Konf. Szkolenovej z Teorii Zagadnien Ekstremalnych*, Lodz. 1990, P. 34–38 (in Polish).
27. V. V. Starkov, Harmonic locally quasiconformal mappings, *Trudy of Petrozavodsk State University. Math.* **1**(1993), 61–69 (in Russian).
28. V. V. Starkov, Harmonic locally quasiconformal mappings, *Annales Universitatis Mariae Curie-Skłodowska. Sectio A. Mathematica,* **14**(1995), 183–197.
29. V. V. Starkov, Univalence disks of harmonic locally quasiconformal mappings and harmonic Bloch functions, *Siberian Math. J.* **38**(1997), 791–800.

S. Yu. Graf, Tver State University, ul. Zhelyabova 33, Tver, 170000 Russia.
E-mail address: sergey.graf@tversu.ru

S. Ponnusamy, Indian Statistical Institute (ISI), Chennai Centre, SETS (Society for Electronic Transactions and Security), MGR Knowledge City, CIT Campus, Taramani, Chennai 600 113, India.
E-mail address: samy@isichennai.res.in, samy@iitm.ac.in

V. V. Starkov, Department of Mathematics, University of Petrozavodsk, ul. Lenina 33, 185910 Petrozavodsk, Russia
E-mail address: vstarv@list.ru